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STRICT 2-THRESHOLD GRAPHS

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A graph G is called a *strict* 2-*threshold* graph if its edge-set can be partitioned into two threshold graphs T_1 and T_2 such that every triangle of G is also a triangle of T_1 or of T_2. We indicate a polynomial-time algorithm to recognize these graphs and characterize them by forbidden configurations.

1. Introduction

All graphs considered are finite, simple and undirected. $V(G)$ and $E(G)$ denote the vertex set and the edge set of a graph G, respectively. We denote by xy an edge with ends x and y, and by $\overline{xy}$ a non-edge with ends x and y. In other words, $\overline{xy}$ indicates that the vertices x and y are not adjacent. If K is any subset of vertices, then $N_K(x)$ denotes the set of all vertices in K that are adjacent to x. The size of $N_K(x)$ is denoted by $\deg_K x$ and is called the *degree* of x in K. A *clique* is a set of pairwise adjacent vertices and a *stable set* is a set of pairwise non-adjacent vertices. A graph induced by a clique is called a *complete graph*, and a complete graph on three vertices is called a *triangle*.

A graph is called a *threshold graph* if there is a hyperplane separating the characteristic vectors of its stable sets from the characteristic vectors of its non-stable sets. In other words, a graph G on n vertices is a threshold graph if there exist reals $a_1, a_2, \dots, a_n$, and t such that the characteristic vectors of the stable sets of G are precisely the 0–1 solutions of

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n \le t.$$

Various characterizations of threshold graphs can be found in [1, 4]. The following two lemmas give two characterizations that are important for our study.

Let K be any subset of vertices of a graph G. We say that the vertices x and y are *comparable in* K if $N_K(x) \subseteq N_K(y) \cup \{y\}$ or $N_K(y) \subseteq N_K(x) \cup \{x\}$.

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Lemma 1.1 [1]. *G is a threshold graph if and only if V(G) can be partitioned into a clique K and a stable set S such that every two vertices in S are comparable in K.*

Lemma 1.2 [1]. *G is a threshold graph if and only if it does not contain any one of the graphs of Fig. 1 as an induced subgraph.*

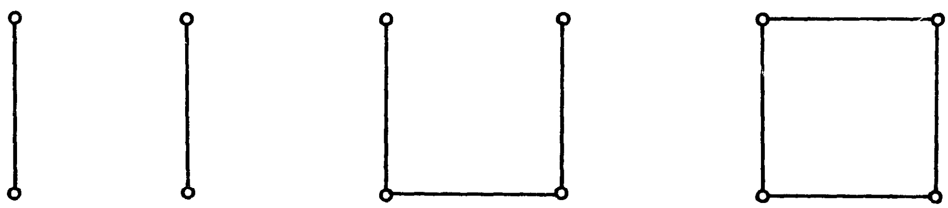


Fig. 1. The forbidden induced subgraphs of threshold graphs.

In all the figures in this paper, solid lines represent edges and broken lines (if any) represent non-edges. A *configuration* is a specification of a set of edges and a set of non-edges of some graph, the two sets being disjoint. A graph G is said to contain a configuration H if all the edges of H are also edges of G and all the non-edges of H are also non-edges of G. For instance, all the graphs on four vertices that contain the configuration of Fig. 2 are shown in Fig. 1. In this terminology, Lemma 1.2 can be restated as follows: G is a threshold graph if and only if it does not contain the configuration of Fig. 2.

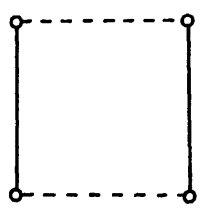


Fig. 2. The forbidden configuration of threshold graphs.

Two non-incident edges e and f are said to be *in conflict* if one end of e is not adjacent to one end of f and the other end of e is not adjacent to the other end of f. For instance, the two edges of Fig. 2 are in conflict. The following remark follows from the above considerations.

G is a threshold graph if and only if no two edges of G are in conflict. (1.1)

The *threshold dimension* of a graph G is the smallest number of threshold subgraphs required to cover $E(G)$. Yannakakis [11] has shown that recognizing the graphs with threshold dimension at most k is NP-complete for any fixed $k \geq 3$. Graphs with threshold dimension 2 are also called *2-threshold graphs*. Cozzens and Leibowitz [3] have reported that recognizing 2-threshold graphs is also NP-complete. For various results on 2-threshold graphs see [2, 5–8, 10].

A graph G is called a *strict 2-threshold graph* if it satisfies the following condition:

$E(G)$ can be partitioned into two threshold graphs T_1 and T_2 such that every clique of G is also a clique of T_1 or of T_2. (1.2)

Strict 2-threshold graphs are known to be comparability graphs [7].

If in (1.2) we replace the phrase "partitioned into" by the phrase "covered by", then we obtain a more general class of graphs called *cobithreshold graphs.* A polynomial-time algorithm to recognize cobithreshold graphs is given in [5], and a partial characterization by forbidden configurations is given in [8] for the complements of these graphs. No complete characterization by forbidden configurations is known for the class of cobithreshold graphs.

In Section 2 we indicate a polynomial-time algorithm to recognize strict 2-threshold graphs and give a list of forbidden configurations for this class. In Section 3 we prove that this list is complete, thus obtaining a characterization of strict 2-threshold graphs by forbidden configurations.

Notice that in the definition (1.2) of strict 2-threshold graphs, we may replace the word "clique" by the word "triangle". In the rest of this paper we use the following equivalent definition for strict 2-threshold graphs:

A graph G is a strict 2-threshold graph if and only if $E(G)$ can be partitioned into two threshold graphs T_1 and T_2 such that every triangle of G is also a triangle of T_1 or of T_2. (1.3)

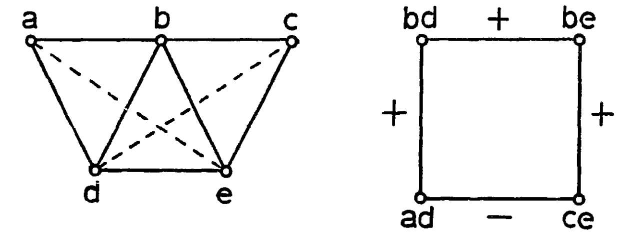
2. Eight forbidden configurations

The algorithm for recognizing strict 2-threshold graphs is a simplified version of the algorithm for recognizing cobithreshold graphs given in [5]. For the sake of completeness, we give here the version of the algorithm for recognizing strict 2-threshold graphs.

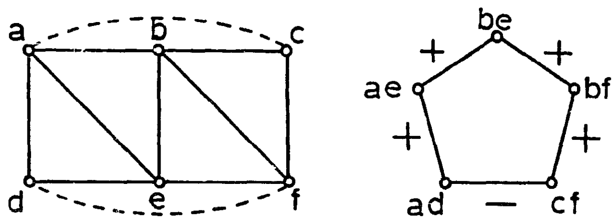
Given any graph G, its *conflict graph* G^* is defined by:

$V(G^*)=E(G)$, and two vertices of G^* are adjacent if and only if the corresponding edges of G are in conflict. (2.1)

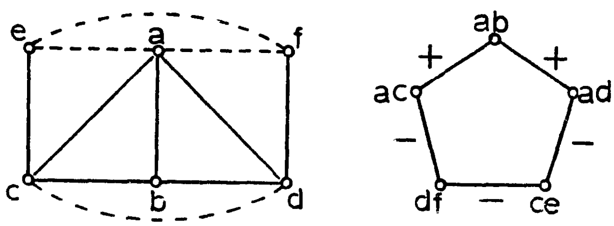
Observe that if T is any threshold subgraph of G, then $E(T)$ corresponds to a stable set of G^*, since by (1.1) no two edges of T are in conflict in G. It follows that a necessary condition for a graph G to be 2-threshold is that G^* is bipartite. Moreover, if T_1 and T_2 are the threshold graphs as given in the definition (1.3) of a strict 2-threshold graph G, then $E(T_1)$ and $E(T_2)$ partition $V(G^*)$ into two stable sets such that the three vertices of G^* corresponding to the edges of any triangle in G are all in the same side of the partition. This gives a necessary condition for a strict 2-threshold graph G. The sufficiency of this condition is stated in Theorem 2.1 below.



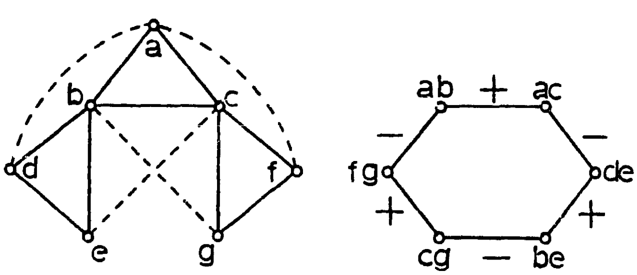
$H_1(abcde)$



$H_2(abcdef)$

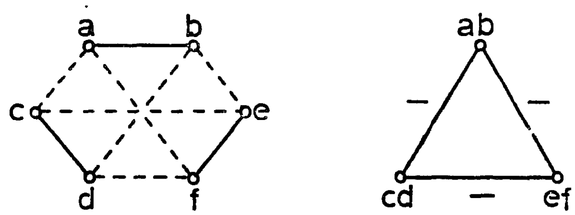


$H_3(abcdef)$

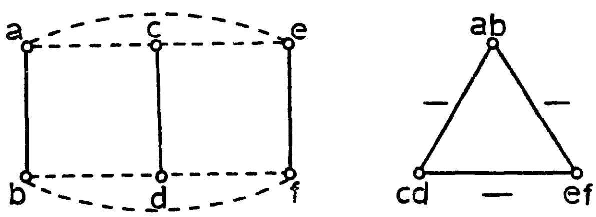


$H_4(abcdefg)$

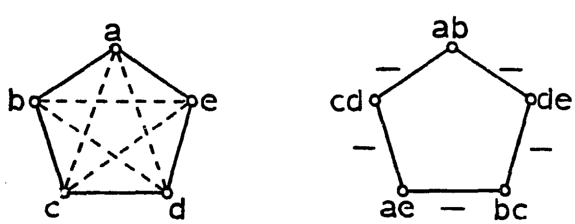
Fig. 3. The eight forbidden configurations of strict 2-threshold graphs. Corresponding negative cycles in the associated signed graphs are also indicated.



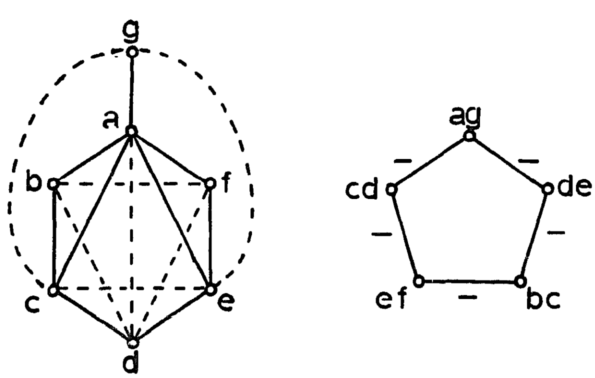
$H_5(abcdef)$



$H_6(abcdef)$



$H_7(abcde)$



$H_8(abcdefg)$

Fig. 3. *Continued.*

A *signed graph* is a graph H in which $E(H)$ is partitioned into two sets $E^+(H)$ and $E^-(H)$, called the sets of positive and negative edges, respectively.

A signed graph H is said to be *balanced* if no cycle of H contains and odd total number of negative edges (a *negative cycle*), or equivalently [9], if $V(H)$ can be partitioned into two sets V_1 and V_2 such that each negative edge of H has an end in V_1 and an end in V_2, and each positive edge of H has both ends in V_1 or both in V_2. (2.2)

Given any graph G, construct its signed graph $H(G)$ as follows:
(1) $V(H)=V(G^*)=E(G)$, $E^-(H)=E(G^*)$ (see (2.1)),
(2) $E^+(H)=\{ef$: e and f are two edges of some triangle in $G\}$. (2.3)

Theorem 2.1. *A graph G is a strict 2-threshold graph if and only if H(G) defined in (2.3) is balanced.*

We omit the proof of this theorem because it is similar to (in fact, is a simplification of) the proof of [5, Theorem 3.7].

The above theorem can be used to recognize strict 2-threshold graphs and find T_1 and T_2 as in (1.3).

Algorithm. Given any graph G with m edges.

(1) Construct its signed graph $H(G)$ given by (2.3). This can be done in $O(m^2)$ time.

(2) Check if $H(G)$ is balanced. This can be done in $O(m^2)$ time.

By Theorem 2.1, G is a strict 2-threshold graph if and only if $H(G)$ is balanced.

The next theorem is useful in characterizing strict 2-threshold graphs by forbidden configurations.

Figure 3 lists eight configurations $H_1, H_2, \ldots, H_8$. To the right of each of them is a negative cycle in the associated signed graph. The configurations H_5 and H_6 are referred to as *triple conflicts* (since they contain three pairwise conflicting edges). The only graph of the configuration H_7 is called a *pentagon*.

Theorem 2.2. *If G is a strict 2-threshold graph, then G does not contain any of the configurations* $H_1, H_2, \ldots, H_8$ *of Fig.* 3.

Proof. It is easy to see that every induced subgraph of a threshold graph is also a threshold graph, and hence every induced subgraph of a strict 2-threshold graph is also a strict 2-threshold graph. Hence it is enough to show that none of $H_1, \ldots, H_8$ can be completed to a strict 2-threshold graph by specifying the missing edges and non-edges. By virtue of Theorem 2.1, this can be done by showing that each of the associated signed graphs of these configurations contains a negative cycle. Such cycles are illustrated in Fig. 3. Hence the theorem. □

3. Sufficiency of forbidding the configurations

We show in this section that every graph that contains none of the configurations listed in Fig. 3 is a strict 2-threshold graph, thus obtaining the promised characterization. We first need to prove a series of lemmas. The following two remarks are easily verified.

Remark 3.1. Two vertices x and y are not comparable in K if and only if there exist two vertices x' and y' in K such that xx', yy', $\overline{xy'}$, and $\overline{yx'}$. We say in this case that *x and y are in conflict in K.*

Remark 3.2. Let e and f be two non-incident edges. Then e and f do not conflict if and only if one end of e is adjacent to both ends of f or one end of f is adjacent to both ends of e. In either case, three of the four ends of e and f form a triangle, and conversely.

Throughout the rest of the paper, let $G=(V,E)$ be a graph not containing any of the configurations shown in Fig. 3. We use the following notations. Let K be a largest clique in G. Put

(1) $\quad L:=V\setminus K,$

(2) $\quad A:=\{x\in L\colon \deg_K x\geq 2\},$

(3) $\quad B:=\{x\in L\colon \deg_K x=1 \text{ and } N_K(x)\subseteq N_K(y) \text{ for all } y\in A\},$

(4) $\quad C:=\{x\in L\setminus B\colon \deg_K x=1\}$, and

(5) $\quad D:=\{x\in L\colon \deg_K x=0\}.$

We refer to an edge between K and A as an *A-edge*, and similarly for *B-edges* and *C-edges*.

Remark 3.3. $N_K(C)\cap N_K(B)$ is empty (where $N_K(X):=\bigcup_{x\in X} N_K(x)$).

Remark 3.4. Every edge in L is in conflict with some edge in K.

Proof. Let ab be an edge in L. Then, K being a largest clique, there exist two distinct vertices a' and b' in K such that $\overline{aa'}$ and $\overline{bb'}$, for otherwise one could enlarge K by adding a or b or both and removing their neighbors in K. Clearly ab and $a'b'$ are in conflict. □

Lemma 3.1. *Every two vertices of A are comparable in K.*

Proof. Assume a,b are two vertices in A that are not comparable in K. Then by Remark 3.1, there exist c,d in K such that ac, bd, $\overline{ad}$ and $\overline{bc}$. If there exists x in K such that xa, xb, then clearly G contains $H_1(axbcd)$—a contradiction. Otherwise there exist x,y in K such that xa, $\overline{xb}$, yb, and $\overline{ya}$. But then G contains $H_2(ycabdx)$—a contradiction. Hence the lemma. □

Lemma 3.2. *The size of $N_K(B \cup C)$ is at most* 2.

Proof. Otherwise G contains a triple conflict. □

Lemma 3.3. *The size of $N_K(C)$ is at most* 1.

Proof. By definition of C, any vertex of C is in conflict in K with some vertex of A (see Remark 3.1). Thus if c_1, c_2 in C have two distinct neighbors in K, then there exist a_1, a_2 in A such that a_i is in conflict in K with c_i for $i=1,2$. If $a_1=a_2$, then G contains a triple conflict. Otherwise a_1 and a_2 are distinct, and we may assume that a_i is not in conflict in K with c_j, $i \neq j$. Then by Remark 3.1, a_1 and a_2 are in conflict in K, contradicting Lemma 3.1. Hence the lemma. □

Lemma 3.4. *The configuration of Fig. 4 is forbidden by our assumptions.*

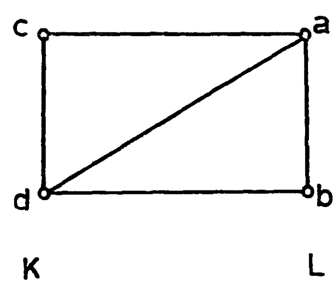


Fig. 4. A forbidden configuration. Vertices c, d are in K and vertices a, b are in L.

Proof. Assume G contains the configuration of Fig. 4. By Remark 3.4, the edge ab in L is in conflict with some edge xy in K with $\overline{ax}$ and $\overline{by}$. If $y=c$, then G contains $H_1(xdbca)$, and otherwise G contains $H_2(acxbdy)$—a contradiction. Hence the lemma. □

The following three corollaries follow from Lemma 3.4.

Corollary 3.4.1. *There are no edges from A to $A \cup B$.*

Corollary 3.4.2. *If both ends of an edge xy in L have a common neighbor in K, then* $\deg_K x = \deg_K y = 1$.

Corollary 3.4.3. *The configuration of Fig. 5 is forbidden.*

Proof. Assume G contains the configuration of Fig. 5. By Corollary 3.4.2, each of the vertices a, b, c, d has only one neighbor in K. Hence if g is any other vertex in K, then G contains $H_4(gefabcd)$—a contradiction. Such g exists because G has triangles. Hence the lemma. □

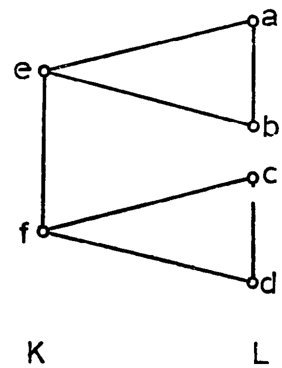


Fig. 5. A forbidden configuration. Vertices e, f are in K and vertices a, b, c, d are in L.

Lemma 3.5. *No two edges of L are in conflict.*

Proof. Assume ab and cd are two edges of L that are in conflict. Also let ef be an edge of K that is in conflict with ab, as shown in Fig. 6. Such an edge exists by Remark 3.4. Now, cd is not in conflict with ef, for otherwise G contains a triple conflict. Hence by Remark 3.2 there exists a triangle on three of the four vertices c, d, e, f.

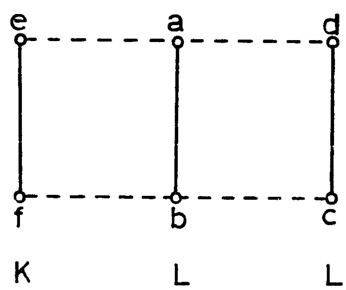


Fig. 6. A configuration in the proof of Lemma 3.5. Vertices e, f are in K and vertices a, b, c, d are in L.

Case 1: cf, df (the case ce, de is similar). By Corollary 3.4.2, $\deg_K c = \deg_K d = 1$. If g is any other vertex in K, which must exist since G has triangles, then bg, else a triple conflict on ab, cd, eg. Therefore $\overline{ag}$, else $H_4(egfabdc)$. Hence be, else a triple conflict on ab, cd, eg. But now G contains $H_3(egfbda)$—a contradiction.

Case 2: ed, fd (the case ec, fc is similar). By Lemma 3.4, $\overline{ec}, \overline{fc}$. If g is any vertex in K not adjacent to d, which must exist by the maximality of K, then bg, else a triple conflict on eg, ab, cd. But now G contains $H_3(fegdbc)$—a contradiction.

Hence the lemma. □

Lemma 3.6. *The configuration of Fig. 7 is forbidden.*

Proof. Assume G contains the configuration of Fig. 7. By Corollary 3.4.2, $\deg_K a = \deg_K b = 1$. By Lemma 3.5, ad, and also ac or bd.

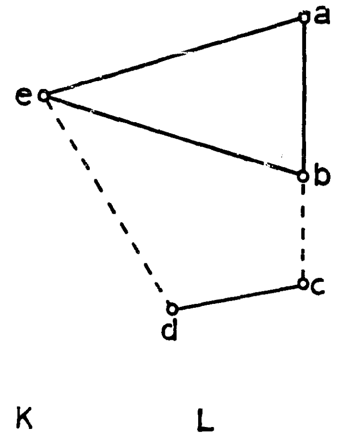


Fig. 7. A forbidden configuration. Vertex e is in K and vertices a, b, c, d are in L.

Case 1: ac. If f, g are any two vertices in K other than e, which must exist since G has triangles, then G contains $H_4(beafgcd)$—a contradiction.

Case 2: bd. If there is a vertex f different from e in K that is not adjacent to c, then G contains $H_3(badecf)$—a contradiction. If not, then c is not adjacent to e and also by the maximality of K, there is a vertex f in K not adjacent to d but adjacent to c. But then c, f, e, b, d form a pentagon—a contradiction.

Hence the lemma. □

Lemma 3.7. *The configuration of Fig.* 8 *is forbidden.*

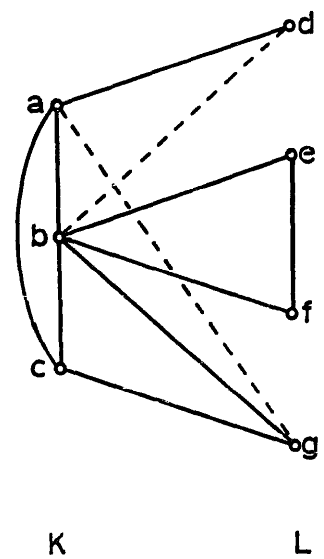


Fig. 8. A forbidden configuration. Vertices a, b, c are in K and vertices d, e, f, g are in L.

Proof. Assume G contains the configuration of Fig. 8. Then $\overline{ea}$, $\overline{ec}$, $\overline{fa}$, $\overline{fc}$, $\overline{eg}$, $\overline{fg}$ by Lemma 3.4. Also by Lemma 3.1, d is not in A, since g is in A and g is not comparable in K to d (see Remark 3.1). Therefore $\overline{dc}$. Hence there is a triple conflict on ad, ef, cg unless ed, df. But then G contains $H_3(efbdga)$—a contradiction. Hence the lemma. □

Lemma 3.8. *If G contains the configuration of Fig.* 9, *then no edge of L conflicts with ce.*

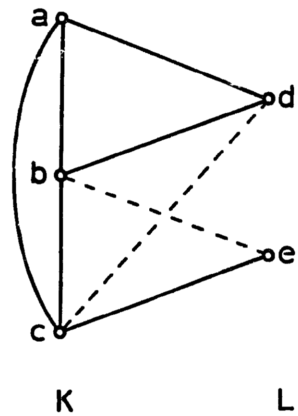


Fig. 9. No edge of L conflicts with ce. Vertices a, b, c are in K and vertices d, e are in L.

Proof. Assume that edge fg in L conflicts with ce due to $\overline{cf}$ and $\overline{eg}$. We have three cases to consider.

Case 1: $f=d$. Clearly $\overline{bg}$ by Lemma 3.4. But then G contains $H_3(bacdeg)$—a contradiction.

Case 2: $g=d$. Clearly $\overline{bf}$ by Lemma 3.4. Therefore ef or else $H_3(bacdef)$. But then G contains a pentagon on b, c, e, f, d—a contradiction.

Case 3: *f, g, are distinct from d.* Edges bd and fg are not in conflict or else a triple conflict on bd, ce, fg. Now by Remark 3.2 there is a triangle on three of the vertices b, d, f, g. Lemma 3.4 rules out bdg and bdf, and Lemma 3.7 rules out bfg. Therefore df, dg and hence $\overline{bg}$ by Lemma 3.4. But then G contains $H_3(bacdeg)$—a contradiction.

Hence the lemma. □

Lemma 3.9. *If G contains the configuration of Fig.* 10, *then no edge of L conflicts with ae.*

Proof. Assume that edge gh in L conflicts with ae due to $\overline{ag}$ and $\overline{eh}$. Notice that $\deg_K x=1$ for $x=c,d$ by Corollary 3.4.2, and for $x=e$ by Lemma 3.7. We have several cases to consider (see Fig. 11).

Case 1: $g=f$. We consider two subcases.

Case 1.1: $h=d$ (the case $h=c$ is similar). We have $\overline{cf}$ else $H_3(dcafeb)$. If t is any other vertex in K, then $\overline{tf}$ else $H_3(tbafed)$. But now G contains $H_8(acdfbte)$—a contradiction.

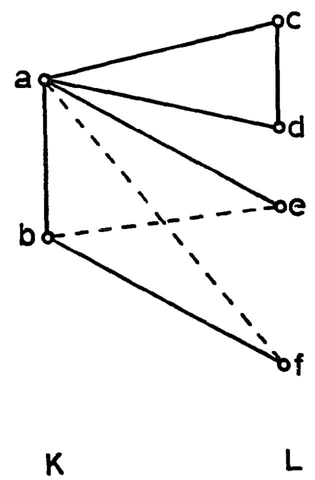


Fig. 10. No edge of L conflicts with ae. Vertices a, b are in K and vertices c, d, e, f are in L.

Case 1.2: *h is distinct from c,d.* By Lemma 3.6 ch, dh, $\overline{ce}$, $\overline{de}$ and then $\overline{ef}$. Also $\overline{fd}$, $\overline{fc}$ or else we are in Case 1.1. Then ah else $H_3(cdahef)$. Therefore $\overline{bh}$ else $H_1(fhdba)$. Let t be a third vertex of K. If tf, then $H_3(btafeh)$, and if $\overline{tf}$, then $H_8(atbfhce)$—a contradiction.

Case 2: $g \neq f$. We consider three subcases.

Case 2.1: $h = f$. We have eg or else we are in Case 1.2. Also $\overline{bg}$ by Corollary 3.4.3. But now $H_7(aegfb)$—a contradiction.

Case 2.2: $h = d$ (the case $h = c$ is similar). We have $\overline{eg}$ by Lemma 3.6, and then $\overline{fg}$ or else we are in Case 1.2. But now there is a triple conflict on dg, ae, bf—a contradiction.

Case 2.3: *h is distinct from c, d, f.* We must have $\overline{fh}$ or else we are in Case 1.2. Now bg or else there is a triple conflict on bf, ae, gh. Then $\overline{fg}$ and $\overline{bh}$ by Corollary 3.4.3. But then there is a triple conflict on gh, ae, bf—a contradiction.

Hence the lemma. □

Lemma 3.10. *If G contains the configuration of Fig.* 12, *then at least one of ac and bd does not conflict with any edge of L.*

Proof. Assume ef and gh are two edges in L such that ef conflicts with ac due to $\overline{ae}$ and $\overline{cf}$, and gh conflicts with bd due to $\overline{bg}$ and $\overline{dh}$. Notice that the edges gh and ef are distinct, or else we have a triple conflict. However, some endpoints of the edges ac, bd, ef, gh may coincide. Thus we have several cases to consider.

Case 1: *ac, bd, ef, gh are pairwise non-incident* (see Fig. 13). To avoid triple conflict on ef, ac, bd and by Remark 3.2 and Lemmas 3.6 and 3.9, we must have de and df, and by symmetry cg and ch. But then ch and df are in conflict, contradicting Lemma 3.5.

Case 2: *ef and gh are incident.* In this case G must contain one of the configurations shown in Fig. 14 or one obtained by symmetry.

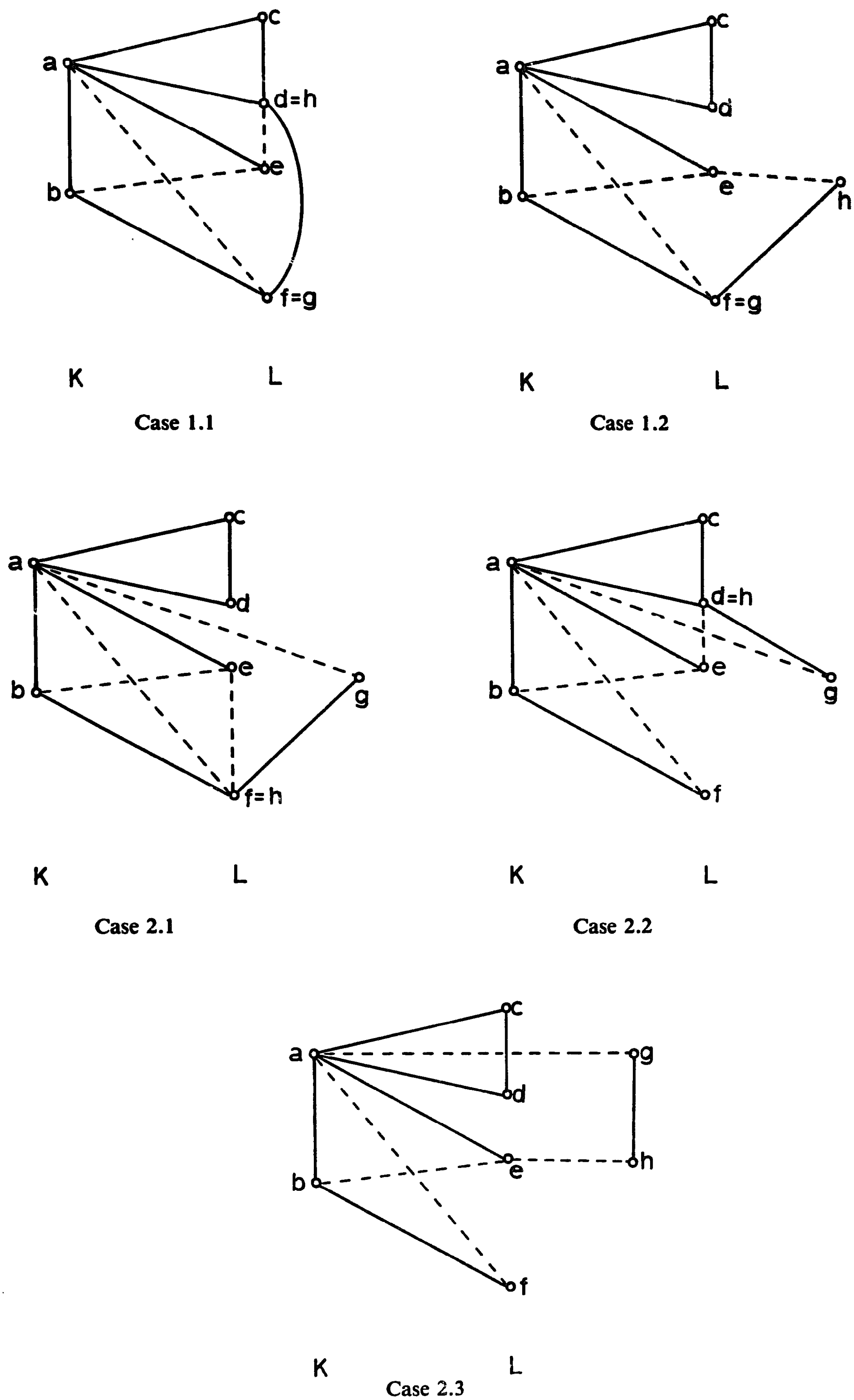


Fig. 11. Cases in the proof of Lemma 3.9. Vertices a, b are in K and vertices c, d, e, f, g, h are in L.

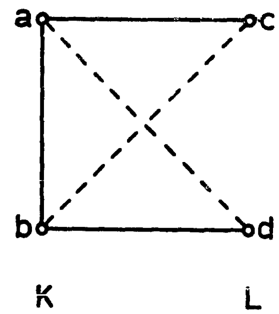


Fig. 12. One of ac, bd does not conflict with any edge of L. Vertices a, b are in K and vertices c, d are in L.

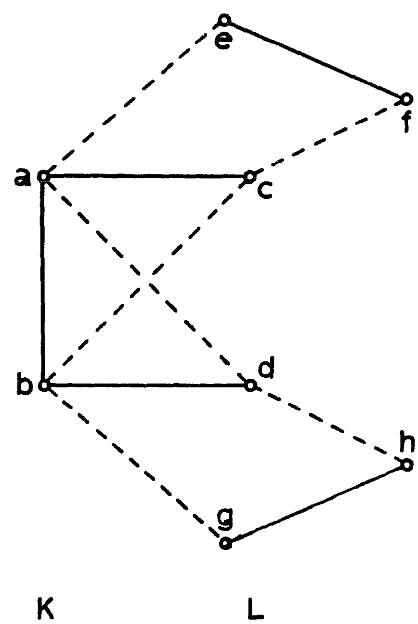


Fig. 13. Case 1 in the proof of Lemma 3.10. Vertices a, b are in K and vertices c, d, e, f, g, h are in L.

Case 2.1. Clearly H contains $H_7(acedb)$—a contradiction.

Case 2.2. We have bf or else a triple conflict on ac, bd, ef. Then $\overline{be}$ by Lemma 3.9 and hence df to avoid a triple conflict on ac, bd, ef. But this contradicts Lemma 3.6.

Case 2.3. We must have de or else we are in Case 2.2. But now $H_7(acedb)$—contradiction.

Case 2.4. We have ch or else a triple conflict on ac, bd, gh. Also by symmetry df. But now ch and df conflict, contradicting Lemma 3.5.

Case 2.5. To avoid triple conflict on ef, ac, bd and by Remark 3.2 and Lemma 3.9, we have be, de. But this contradicts Lemma 3.6.

Case 2.6. To avoid a triple conflict on ac, bd, fh and by Remark 3.2 and Lemma 3.6, we must have af, ah. But this contradicts Lemma 3.9.

Case 3: *ef and gh are not incident, but not Case* 1. In this case G contains one of the configurations shown in Fig. 15 or a symmetric one.

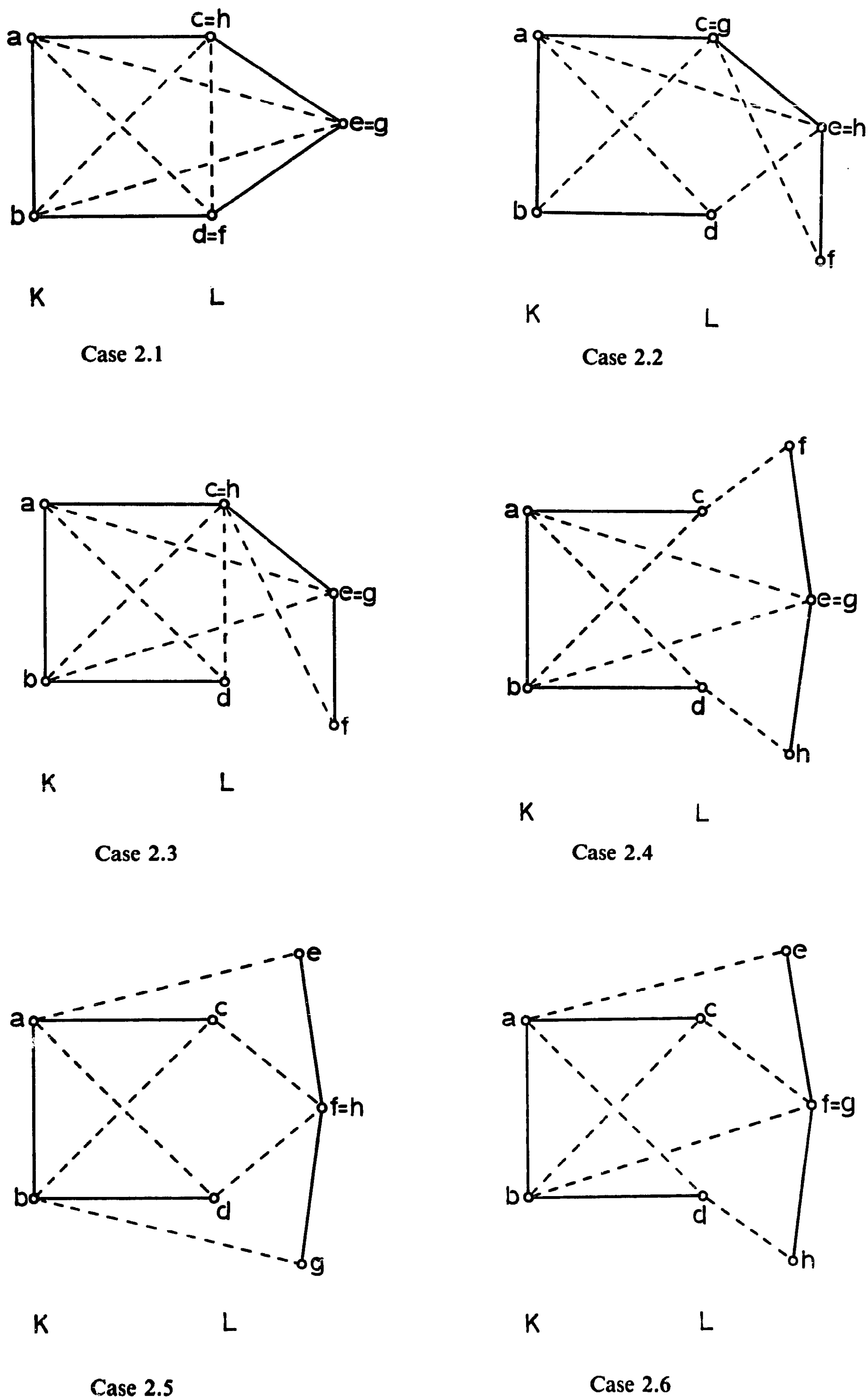
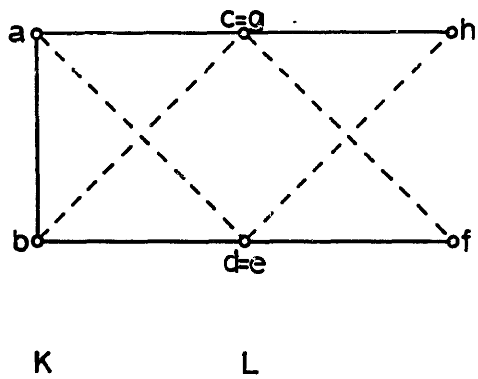
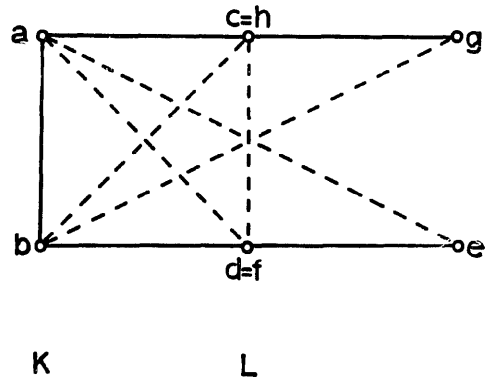


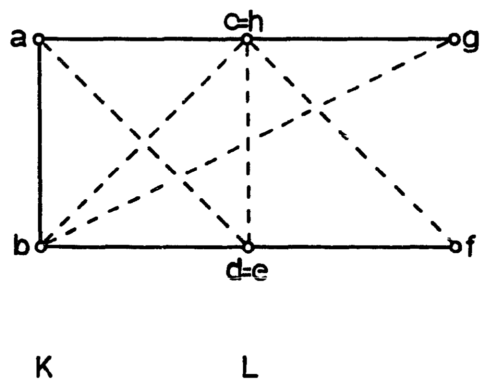
Fig. 14. Case 2 in the proof of Lemma 3.10. Vertices a, b are in K and vertices c, d, e, f, g, h are in L.



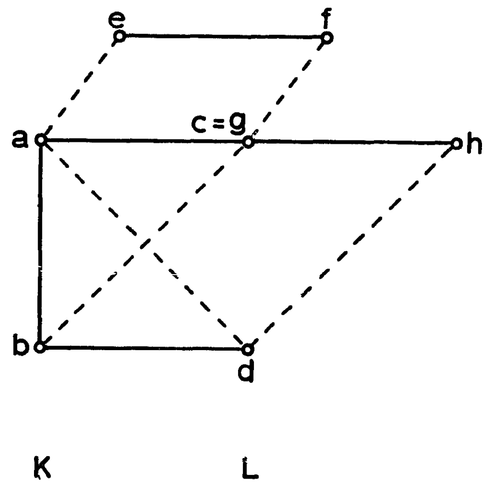
Case 3.1



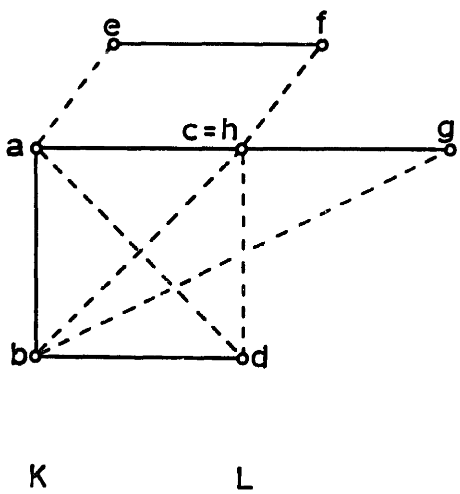
Case 3.2



Case 3.3



Case 3.4



Case 3.5

Fig. 15. Case 3 in the proof of Lemma 3.10. Vertices a, b are in K and vertices c, d, e, f, g, h are in L.

Case 3.1. Two edges in L conflict, contradicting Lemma 3.5.

Case 3.2. We have dg or ce or else we are in Case 3.1. Assume without loss of generality dg. Now ag or else $H_7(acgdb)$. But this contradicts Lemma 3.6.

Case 3.3. We have $\overline{ag}$ by Lemma 3.6. Also dg to avoid Case 3.1. But then $H_7(acgdb)$—a contradiction.

Case 3.4. To avoid a triple conflict on ef, ac, bd and by Remark 3.2 and Lemmas 3.6 and 3.9, we have de, df. But now edges df and ch are in conflict, contradicting Lemma 3.5.

Case 3.5. We have dg or else we are in Case 3.4. Also $\overline{ag}$ by Lemma 3.6. But now $H_7(acgdb)$—a contradiction.

Hence the lemma. □

We now show, using the above ten lemmas, that if G does not contain any of the configurations of Fig. 3, then G is a strict 2-threshold graph. That is, by (1.3), $E(G)$ can be partitioned into two colors, say red and blue, so that (i) each color induces a threshold graph, and (ii) no triangle of G is mixed (we say that a triangle is *mixed* if its edge set contains both colors).

Notice that by Lemma 3.2, the size of $N_K(B)$ is at most 2. We consider all possible values of $|N_K(B)|$ and show in each case that G is a strict 2-threshold graph.

Case 1: $|N_K(B)|=0$.

Lemma 3.11. *In Case* 1, *G is a strict 2-threshold graph.*

Proof. Since $N_K(B)$ is empty, B is also empty. Color the edges of the graph induced by $K \cup A$ red and all other edges blue.

K is a clique and A is a stable set by Corollary 3.4.1. Further, any two vertices of A are comparable in K by Lemma 3.1. Hence the red edges form a threshold graph by Lemma 1.1.

Each blue edge is either a C-edge or an edge entirely in L. Two edges in L do not conflict by Lemma 3.5. Two C-edges cannot conflict, since they are incident in K by Lemma 3.3. Further, an edge in L cannot conflict with a C-edge by Lemma 3.8. Thus no two blue edges are in conflict, and hence by (1.1) the blue edges form a threshold graph as well.

Finally, assume there is a mixed triangle on vertices a, b, c with ab red. At least one of the two vertices a, b is in K. Clearly c is in C. If both a and b are in K, then $\deg_K c=2$, which is impossible by the definition of C. Assume without loss of generality that a is in K and b is in A. Then by Corollary 3.4.2, $\deg_K b=1$, contradicting the definition of A. Thus there are no mixed triangles, proving the lemma. □

Case 2: $|N_K(B)|=1$.

Lemma 3.12. *In Case 2, G is a strict 2-threshold graph.*

Proof. Partition B into two sets P, Q as follows:

$$P:=\{x\in B\colon x \text{ is isolated in } G[B]\}$$

(here $G[B]$ denotes the graph induced by the vertices in B),

$$Q:=B\setminus P.$$

Color the edges of the graph induced by $K\cup A\cup P$ red and all the other edges blue.

$A\cup P$ is a stable set by Corollary 3.4.1 and further, any two vertices in $A\cup P$ are comparable in K by Lemma 3.1, the definition of B, and the condition of Case 2. Hence again by Lemma 1.1, the red edges form a threshold graph.

To show that the blue edges form a threshold graph, we distinguish between two cases.

Case 2.1: *Q is empty.* In this case, as in Lemma 3.11, the blue edges form a threshold graph.

Case 2.2: *Q is not empty.* In this case, by Lemma 3.7, C is empty. Once again, no two edges in L are in conflict by Lemma 3.5. No two B-edges can conflict, since they are incident in K by the condition of Case 2. Finally, a B-edge with one end in Q cannot conflict with an edge in L by Lemma 3.6. Hence the blue edges form a threshold graph as well.

Now assume G contains a mixed triangle on vertices a, b, c with ab red. Clearly then $\{a,b\}\subseteq K\cup A\cup P$ and $c\notin K\cup A\cup P$. Now if $\{a,b\}\subseteq K$, then $\deg_K c\geq 2$, and hence $c\in A$ - a contradiction. Also $\{a,b\}\nsubseteq A\cup P$, since $A\cup P$ is a stable set by Corollary 3.4.1. Therefore assume without loss of generality that $a\in K$ and $b\in A\cup P$. Then $c\notin Q$, hence $c\in C$ by Corollary 3.4.1. But now by Corollary 3.4.2, $\deg_K b=1$ and hence $b\in P$. Therefore the vertices b and c have a common neighbor a in K, contradicting Remark 3.3. It follows that G has no mixed triangles, hence the lemma. □

Case 3: $|N_K(B)|=2$.

Lemma 3.13. *In Case 3, G is a strict 2-threshold graph.*

Proof. First, by Lemma 3.2 and Remark 3.3, it follows that C is empty. Let $N_K(B)=\{r,s\}$ and partition B into two non-empty sets R and S as follows:

$$R:=\{x\in B\colon x \text{ is adjacent to } r\},$$

$$S:=\{x\in B\colon x \text{ is adjacent to } s\}.$$

By Corollary 3.4.3, it follows that at least one of R and S is a stable set. We distinguish between two cases.

Case 3.1: *Both R and S are stable sets.* By virtue of Lemma 3.10, we may assume without loss of generality that no edge in L is in conflict with any edge from K to

S. Color the edges induced by $K \cup A \cup R$ red and all the other edges blue. It is easy to show that each color forms a threshold graph: for the red edges as in Lemma 3.12 and for the blue edges by Lemma 3.5 and our recent assumption.

Assume G contains a mixed triangle on vertices a, b, c, with ab red. Again, as in Lemma 3.12, we may assume without loss of generality that a is in K, b is in $A \cup R$ and c is in S. Hence by Corollary 3.4.2, $\deg_K b = \deg_K c = 1$, and hence $b \in R$. But then b and c have a common neighbor a in K, contradicting the definitions of R and S. Hence G does not contain any mixed triangles, proving the lemma in this case.

Case 3.2: *Only one of R and S is a stable set, say R.* Observe that in this case, no edge of L can be in conflict with any edge from K to S by Lemmas 3.6 and 3.9. The rest of the argument is the same as in Case 3.1. Hence the lemma. □

Theorem 3.14. *If G does not contain any of the configurations of Fig.* 3, *then G is a strict* 2*-threshold graph.*

Proof. Follows from Lemmas 3.11, 3.12 and 3.13. □

Theorem 3.15. *A graph G is a strict* 2*-threshold graph if and only if G does not contain any of the configurations $H_1, H_2, \ldots, H_8$ of Fig.* 3.

Proof. Follows from Theorems 2.2 and 3.14. □

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